

A study of separability criteria for mixed three-qubit states

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We study the noisy GHZ-W mixture. We demonstrate some necessary but not sufficient criteria for different classes of separability of these states. It turns out that the partial transposition criterion of Peres [1] and the criteria of Gühne and Seevinck [2] dealing with matrix elements are the strongest ones for different separability classes of this 2 parameter state. As a new result we determine a set of entangled states of positive partial transpose.

I. INTRODUCTION

The notion of entanglement [3] is the characteristic trait of quantum mechanics. It serves as a resource for Quantum Information Theory [4], a relatively new field of science dealing with the properties, characterisation and applications (e.g. quantum computation [4], quantum teleportation [5]) of the nonlocal behavior of entangled quantum states. A variety of methods of Quantum Information Theory uses pure entangled states of a quantum system which can be easily prepared and which are easy to use to get nonclassical results. However, in a laboratory one can not get rid of the interaction with the environment perfectly, thus the separable compound state of the system and the environment evolves into an entangled one, the prepared pure state of the system evolves into a mixed one.

Generally it is a difficult question to decide whether a mixed state is entangled or not. [3, 6] A density operator representing the state acting on an $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$ composite Hilbert space by definition is *separable* [7] when it can be written as a convex combination of products of local density operators, i.e. if there exists a decomposition of the form

$$\varrho = \sum_i p_i \varrho_i^A \otimes \varrho_i^B, \quad (1)$$

where $0 \leq p_i$, $\sum_i p_i = 1$, and ϱ_i^A and ϱ_i^B are positive operators of trace one acting on subsystems \mathcal{H}^A and \mathcal{H}^B respectively. Classical correlations can give rise only to separable states in the sense of Eq. (1). [7]

A decomposition like the one of Eq. (1) is not unique, and it is difficult to decide whether for a given density operator such a decomposition exists at all. One can make some observations for separable pure states which can be extended to mixed states with the help of convex calculus. The *separability criteria* obtained in this way are *necessary but not sufficient* ones. (Or equivalently sufficient but not necessary *criteria of entanglement*.) On the other hand one can construct *necessary and sufficient* criteria using sophisticated mathematical

methods. [8] Unfortunately these criteria are difficult to use for general density matrices and only the necessary but not sufficient criteria can be used in practice. [9] In this paper we calculate explicitly some of the necessary but not sufficient criteria of separability for a particular two parameter mixture of *three-qubit* density matrices. The form of these density matrices are simple enough to calculate explicitly the set of states for which these criteria hold.

The organization of this paper is as follows. First of all in Section II we briefly review the *separability classes of three-qubit mixed states* using the notions α_k -separability and k -separability. In Section III we introduce our parametrized permutation-invariant family of *three-qubit* density matrices and make some observations about the separability class structure of *permutation-invariant* three-qubit mixed states. After having set the stage, in the next sections we investigate some criteria for separability classes. First in Section IV we consider our quantum-state as a 2×4 qubit-qudit system and we recall and use some *bipartite* separability criteria, namely the *majorization* and the *entropy criteria* related to the notion of mixedness of the subsystems (Sections IV A and IV B respectively), the *partial transposition* and the *reduction criteria* which are particular cases of the positive map criterion (Sections IV C and IV D respectively), and the *reshuffling criterion* which in addition to the partial transposition criterion is the other one of the two independent permutation criteria for two-partite systems (Section IV E). As a next step in Section V we consider our quantum-state as a proper $2 \times 2 \times 2$ three-qubit system and investigate some *three-partite* criteria for separability classes. In Section V A we recall the *permutation criteria* for permutation-invariant three-qubit case giving rise to *another reshuffling criterion*. Then we use some criteria using the *expectation value of local spin-observables* (Section V B), *swap operators* (Section V C) and *explicit expressions of matrix elements* (Section V D). The latter makes it possible to determine a set of entangled states of positive partial transpose. In Section VI we investigate the *SLOCC classes of fully entangled states*. A *summary* is given in Section VII. The *explicit form* of the corresponding matrices, some examples for permutation-invariant states of special separability classes and a detailed calculation of *Wootters-concurrence* of the corresponding two-qubit subsystems are left to Appendices A,

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B and C respectively.

II. SEPARABILITY CLASSES

A two-partite mixed state can be either separable or entangled, depending on the existence of a decomposition as given by Eq. (1). However, the structure of separability classes can be very complex even for three subsystems. To get the adequate generalization of Eq. (1) we recall the definitions of k -separability and α_k -separability as given in Ref. [10].

Consider an N -qubit system with Hilbert space $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2 \otimes \dots \otimes \mathcal{H}^N$ and denote the full set of states for this system as \mathcal{D}_N . Let $\alpha_k = (S_1, S_2, \dots, S_k)$ denote a *partition* of the labels $\{1, 2, \dots, N\}$ into $k \leq N$ disjoint nonempty subsets S_n . A density matrix is α_k -separable, i.e. separable under the *particular* k -partite split α_k , if and only if it can be written as a convex combination of product states with respect to the split α_k . We denote the set of these states as $\mathcal{D}_N^{\alpha_k}$:

$$\varrho \in \mathcal{D}_N^{\alpha_k} : \quad \varrho = \sum_i p_i \otimes_{n=1}^k \varrho_i^{S_n}, \quad (2)$$

where $0 \leq p_i$, $\sum_i p_i = 1$ and $\varrho_i^{S_n}$ is a density operator of the subsystem corresponding to S_n in the split α_k (i.e. acting on $\mathcal{H}^{S_n} = \otimes_{a \in S_n} \mathcal{H}^a$). More generally, for a given k we can consider states which can be written as a mixture of $\alpha_k^{(i)}$ -separable states for generally different $\alpha_k^{(i)}$ splits. These states are called k -separable states and denoted as $\mathcal{D}_N^{k\text{-sep}}$:

$$\varrho \in \mathcal{D}_N^{k\text{-sep}} : \quad \varrho = \sum_i p_i \otimes_{n=1}^k \varrho_i^{S_n^{(i)}}, \quad (3)$$

where $0 \leq p_i$, $\sum_i p_i = 1$ and $\varrho_i^{S_n^{(i)}}$ is a state of the subsystem corresponding to $S_n^{(i)}$ in the split $\alpha_k^{(i)}$ and in this case the $\alpha_k^{(i)} = (S_1^{(i)}, S_2^{(i)}, \dots, S_k^{(i)})$ k -partite splits can be different for different i .

Clearly $\mathcal{D}_N^{k+1\text{-sep}} \subset \mathcal{D}_N^{k\text{-sep}}$, so the notion of k -separability gives rise to a natural hierarchic ordering of the states. The full set of states is $\mathcal{D}_N = \mathcal{D}_N^{1\text{-sep}}$ and we call elements of $\mathcal{D}_N^{k\text{-sep}} \setminus \mathcal{D}_N^{k+1\text{-sep}}$ (i.e. the k -separable but not $k+1$ separable states) “ k -separable entangled”. We call the N -separable states *fully separable*, the 2-separable states *biseparable* and the 1-separable entangled states *fully entangled*.

Clearly, $\mathcal{D}_N^{\alpha_k}$ is a convex set, and so is $\mathcal{D}_N^{k\text{-sep}}$, because it is the convex hull of the union of $\mathcal{D}_N^{\alpha_k}$ s for a given k . Note that these definitions allow a k -separable state not to be α_k -separable for any particular split α_k , and a state which is α_k -separable for all α_k partitions not to be $k+1$ -separable. The existence of such states is somehow counterintuitive, but explicit examples for these states can be found in literature. (Using a method dealing with *Unextendible Product Bases* Bennett *et. al.* have constructed

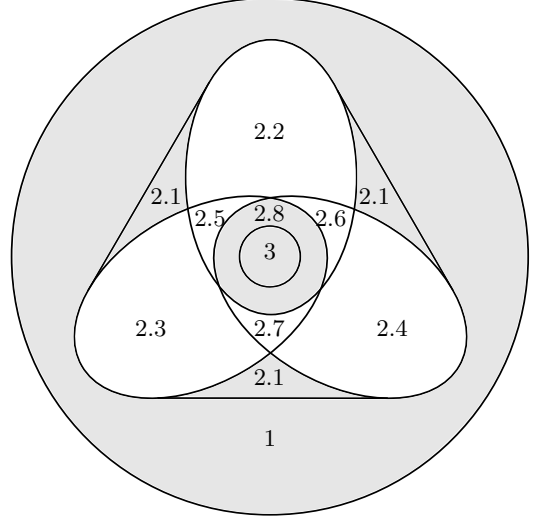


FIG. 1. Separability classes for three qubits. The tinted subsets of the diagram can contain permutation-invariant states.

a three-qubit state which is separable for all α_2 but not fully separable [11]. Another three-qubit example can be found in Ref. [12].)

Let us now consider the three-qubit case. (We adopt the notations of [10].) For three qubits we have the partitions: $\alpha_1 = 123$, $\alpha_2 = 1 - 23$, $\alpha_2 = 2 - 31$, $\alpha_2 = 3 - 12$, $\alpha_3 = 1 - 2 - 3$ (here we use a simplified notation for partitions usual in literature). With this, the *partial separability classes* of mixed three-qubit states are as follows. (See also in Ref. [10] and in Fig. 1.)

Class 3: This is the set of fully separable three-qubit states: $\mathcal{D}_3^{3\text{-sep}} = \mathcal{D}_3^{1-2-3}$.

Classes 2.1–2.8: These are the disjoint subsets of 2-separable entangled states $\mathcal{D}_3^{2\text{-sep}} \setminus \mathcal{D}_3^{3\text{-sep}}$. Classes 2.2–2.8 can be obtained by the set-theoretical intersections of \mathcal{D}_3^{1-23} , \mathcal{D}_3^{2-31} and \mathcal{D}_3^{3-12} . (See in Fig. 1.) For example *Class 2.8* is $(\mathcal{D}_3^{1-23} \cap \mathcal{D}_3^{2-31} \cap \mathcal{D}_3^{3-12}) \setminus \mathcal{D}_3^{1-2-3}$ (i.e. states that can be written as convex combination of 3 – 12-separable states and can also be written as convex combination of 2 – 31-separable states and can also be written as convex combination of 1 – 23-separable states but can not be written as convex combination of 1 – 2 – 3-separable states), *Class 2.7* is $(\mathcal{D}_3^{2-31} \cap \mathcal{D}_3^{3-12}) \setminus \mathcal{D}_3^{1-23}$ (i.e. states that can be written as convex combination of 3 – 12-separable states and can also be written as convex combination of 2 – 31-separable states but can not be written as convex combination of 1 – 23-separable states), *Class 2.2* is $\mathcal{D}_3^{1-23} \setminus (\mathcal{D}_3^{2-31} \cup \mathcal{D}_3^{3-12})$ (i.e. states that can be written as convex combination of 1 – 23-separable states but can not be written as convex combination of 2 – 31 or 3 – 12-separable states). On the other hand the union of the sets of α_2 -separable states is not a convex one, it is a genuine subset of its convex hull $\mathcal{D}_3^{2\text{-sep}}$. This defines *Class 2.1* as $\mathcal{D}_3^{2\text{-sep}} \setminus (\mathcal{D}_3^{1-23} \cup \mathcal{D}_3^{2-31} \cup \mathcal{D}_3^{3-12})$, i.e. states that are 2-separable but can not be written as convex combination of α_2 -separable states for any partic-

ular α_2 . However, we do not consider these states fully entangled since they can be mixed without the use of genuine three-partite entanglement.

Class 1: This contains all the fully entangled states of the system: $\mathcal{D}_3^{1\text{-sep}} \setminus \mathcal{D}_3^{2\text{-sep}}$.

III. A SYMMETRIC FAMILY OF MIXED THREE-QUBIT STATES

Let ϱ be the mixture of the Greenberger-Horne-Zeilinger state, the W state and the maximally mixed three-qubit state:

$$\varrho = d\frac{1}{8}\mathbb{I} + g|\text{GHZ}\rangle\langle\text{GHZ}| + w|\text{W}\rangle\langle\text{W}|, \quad (4)$$

where $0 \leq d, g, w \leq 1$ are real numbers giving rise to the probability distribution characterizing the mixture, i.e. $d + g + w = 1$. (In the following sections we plot the subsets of states for which the separability criteria hold on the g - w -plane, i.e. we project the probability-simplex onto the $d = 0$ plane. A point on this plane determines the third coordinate: $d = 1 - g - w$. Sometimes it is instructive to use the renormalized parameters $\tilde{d} = d/8$, $\tilde{g} = g/2$, $\tilde{w} = w/3$.) In Eq. (4) \mathbb{I} denotes the 8×8 identity matrix and the usual GHZ and W states are

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \quad (5a)$$

$$|\text{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \quad (5b)$$

These two states are representative elements of the two different *SLOCC-classes* [13] of genuine-entangled three-qubit states. [14] The *GHZ state* is maximally entangled in the sense that its one-partite subsystems are maximally mixed. On the other hand, its two-partite subsystems are separable (having diagonal density matrix). The one-partite subsystems of the *W-state* are less mixed than the ones of the GHZ state, but its two-partite subsystems are entangled with Wootters-concurrence 2/3. (See Appendix C.)

The *GHZ-W mixture* ($d = 0$ line) is well studied: the three tangle [15] with its convex roofs [16], the Wootters-concurrences [17, 18], the one tangle and the mixed-state CKW-inequality [15] were given for this mixture in the paper of Lohmayer *et. al.* [19]. These results give an upper bound for values of these quantities on the whole simplex defined in Eq. (4): if $f(\varrho_{g,w}) = \min \sum_i p_i f(\psi_i)$ where the minimum is taken over all decomposition $\sum_i p_i |\psi_i\rangle\langle\psi_i| = \varrho_{g,w}$, and $\varrho_{d,g,w} = d\mathbb{I}/8 + (1-d)\varrho_{g/(1-d),w/(1-d)}$, and $f(\psi) = 0$ on product states then $f(\varrho_{d,g,w}) \leq (1-d)f(\varrho_{g/(1-d),w/(1-d)})$.

The *maximally mixed three-qubit state* can be regarded in some sense as the “center” of the set of density matrices. On the other hand this state is sometimes called “white noise” because of its uniform spectrum. Mixing a state with white noise is the way to investigate the effect

of environmental decoherence. [6] A noisy state is usually of full rank, so methods for density matrices of low rank (like range criterion [20], or finding optimal decompositions with respect some pure-state measures) usually fail for such states.

On the other hand, there are exact results for the *GHZ-white noise mixture* ($w = 0$ line). In Ref. [21] Dür and Cirac, using their results about a special class of GHZ-diagonal states have shown that ϱ is fully separable *if and only if* $0 \leq g \leq 1/5$. Moreover, it follows from their observations that if the state is separable under a bipartition then it is fully separable, so Class 2.8 is empty for these states. In Ref. [2] Gühne and Seevinck gives *necessary and sufficient* condition of genuine three-partite entanglement for GHZ-diagonal states, which contain the noisy GHZ state: for $1/5 < g \leq 3/7$ the state is biseparable, yet inseparable under bipartitions, i.e. in Class 2.1, and for $3/7 < g \leq 1$ the state is fully entangled. Unfortunately there are no such results for other subsets of the simplex given in Eq. (4).

The noisy GHZ-W mixture given in Eq. (4) is clearly a permutation invariant one, hence the reduced density matrices of ϱ are all of the same form: $\varrho^{12} = \varrho^{23} = \varrho^{31}$ and $\varrho^1 = \varrho^2 = \varrho^3$, where $\varrho^{12} = \text{Tr}_3 \varrho$, $\varrho^1 = \text{Tr}_{23} \varrho$ and so forth. The explicit forms of these matrices are given in Eqs. (A3) and (A4) of the Appendix.

What can we say about the separability-classes of Section II for permutation-invariant three-qubit states *in general*? Clearly, if a permutation-invariant state is in $\mathcal{D}_3^{\alpha_2}$ for a *particular* α_2 , then it is in $\mathcal{D}_3^{\alpha_2}$ for *every* α_2 . So permutation-invariant states can not be in Classes 2.2-2.7, we have to investigate separability criteria only for Class 2.1, Class 2.8 and Class 3. (Fig. 1.)

(Note that the biseparability of a permutation-invariant state does not mean that the decomposition of Eq. (3) contains only permutation-invariant members, since if the latter holds then the state must be the white noise. To see this, write out a member of the decomposition with the help of the σ_i Pauli-matrices and x_i, y_j real coefficients as $\varrho^1 \otimes \varrho^{23} = \frac{1}{2}(\mathbb{I} + \sum_i x_i \sigma_i) \otimes \frac{1}{4}(\mathbb{I} \otimes \mathbb{I} + \sum_j y_j \sigma_j \otimes \sigma_j) = \frac{1}{8}(\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} + \sum_i x_i \sigma_i \otimes \mathbb{I} \otimes \mathbb{I} + \sum_j y_j \mathbb{I} \otimes \sigma_j \otimes \sigma_j + \sum_{ij} x_i y_j \sigma_i \otimes \sigma_j \otimes \sigma_j)$ which can be permutation-invariant if and only if $x_i = 0, y_j = 0$. The reverse of this is that for permutation-invariant states in Classes 2.1 and 2.8 there does not exist a decomposition as in Eq. (3) containing *only* permutation-invariant members.)

The remaining question is whether the remaining classes can contain permutation-invariant states in general. Class 1 and Class 3 is clearly nonempty for permutation-invariant states, and for Classes 2.1 and 2.8 we show explicit examples in Appendix B.

If we consider the $2 \times 2 \times 2$ three-qubit system as a 2×4 qubit-qudit system then some well-known and easy-to-use two-partite separability criteria give rise to separability criteria for $\bigcap_{\alpha_2} \mathcal{D}_3^{\alpha_2}$, hence for the union of Class 2.8 and Class 3. (This one is also a convex set since it is the intersection of convex ones.) First we investigate these criteria.

IV. TWO-PARTITE SEPARABILITY CRITERIA

In this section we consider our system as a 2×4 qubit-qudit system (with Hilbert-spaces $\mathcal{H}^A = \mathcal{H}^1$ and $\mathcal{H}^B = \mathcal{H}^{23}$) and investigate some criteria of 1 – 23-separability which means the union of Classes 2.8 and 3. To do this we will need the spectra of the density matrix ϱ given in Eq. (4) and its marginals. (The explicit forms of these matrices are given in Eqs. (A1), (A3) and (A4) of the Appendix.) Due to the special structure of ϱ finding the eigenvalues of these matrices is not a difficult task. It turns out that all of the relevant eigenvalues are linear in the parameters g and w :

$$\begin{aligned} \text{Spect}(\varrho) = \{ & \tilde{d} + 2\tilde{g} = (3 + 21g - 3w)/24, \\ & \tilde{d} + 3\tilde{w} = (3 - 3g + 21w)/24, \\ & \tilde{d} = (3 - 3g - 3w)/24 \quad 6 \text{ times} \}, \end{aligned} \quad (6a)$$

$$\begin{aligned} \text{Spect}(\varrho^{23}) = \{ & 2\tilde{d} + 2\tilde{w} = (6 - 6g + 10w)/24, \\ & 2\tilde{d} + \tilde{g} + \tilde{w} = (6 + 6g + 2w)/24, \\ & 2\tilde{d} + \tilde{g} = (6 + 6g - 6w)/24, \\ & 2\tilde{d} = (6 - 6g - 6w)/24 \quad \}, \end{aligned} \quad (6b)$$

$$\begin{aligned} \text{Spect}(\varrho^1) = \{ & 4\tilde{d} + \tilde{g} + 2\tilde{w} = (12 + 4w)/24 \\ & 4\tilde{d} + \tilde{g} + \tilde{w} = (12 - 4w)/24 \}. \end{aligned} \quad (6c)$$

Here and in the following, we give expressions with and without \sim . This is because the expressions with the quantities $\tilde{d}, \tilde{g}, \tilde{w}$ can be expressive as they refer to the original mixing weights, on the other hand we plot in the g, w coordinates.

A. Majorization criterion

First of all we invoke the notion of *majorization for probability distributions*. Let $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ be two probability distributions of length n . The definition of majorization is as follows. First we order p and q in non-increasing order (we denote this as p^\downarrow). Then p is majorized by q by definition when the following inequality holds for all k :

$$\sum_{i=1}^k p_i^\downarrow \leq \sum_{i=1}^k q_i^\downarrow, \quad 1 \leq k \leq n. \quad (7)$$

This is denoted by $p \prec q$. (For $k = n$ the inequality turns to equality since both sides of it are equal to 1. If the length of p and q differs, one can add some zeroes to the shorter one.)

The majorization is clearly *reflexive* ($p \prec p$) and *transitive* (if $p \prec q$ and $q \prec r$ then $p \prec r$) but the *antisymmetry* (if $p \prec q$ and $q \prec p$ then $p = q$) holds only in a restricted manner: if $p \prec q$ and $q \prec p$ then $p^\downarrow = q^\downarrow$. Hence the majorization defines a *partial order* on the set of probability distributions *up to permutations*. It is clear that $p \not\prec q$ does not imply $q \prec p$, in other words there exist pairs of probability distributions which we can not compare by majorization. (For example let $p^\downarrow := (1/2, 1/8, \dots)$ and $q^\downarrow := (1/3, 1/3, \dots)$, then $p \not\prec q$ and $q \not\prec p$.) This is why the majorization gives rise merely to a *partial* ordering.

With respect to majorization the set of probability distributions contains a greatest and a smallest element. One can check that all probability distribution p majorize the uniform distribution and p is majorized by the distribution containing only one element: $(\frac{1}{n}, \frac{1}{n}, \dots) \prec p \prec (1, 0, \dots)$. Employing the notion of majorization we can compare the amount of disorder contained in different probability distributions. If $p \prec q$ then we can say that p is more disordered than q or equivalently q is more ordered than p , but there are pairs of distributions for which their rank of disorder can not be compared.

The *majorization of density matrices* is defined via the corresponding majorization of their spectra. Let ω and σ be two density matrices, then $\omega \prec \sigma$ by definition when

$$\text{Spect}(\omega) \prec \text{Spect}(\sigma). \quad (8)$$

Now we can turn to the *majorization criterion* for two-partite systems. It has been found by Nielsen and Kempe [22], and it states that *for a separable state the whole system is more disordered than any of its subsystems*:

$$\varrho \text{ separable} \implies \varrho \prec \varrho^A \text{ and } \varrho \prec \varrho^B. \quad (9)$$

The rhs. of (9) can also be true for entangled states, but if it does not hold then the state must be entangled.

Let us see what the majorization criterion states about the noisy GHZ-W mixture ϱ given by Eq. (4). We can write out the rhs. of (9) explicitly using the spectra given by Eqs. (6a)-(6c), then we have to decide whether the inequalities in (7) hold. For this we have to order the eigenvalues of the density matrices in non-increasing order. These orderings depend on the ranges of the parameters and it turns out that we have to distinguish between four cases. These cases are as follows: $0 \leq g \leq \frac{2}{3}w$, $\frac{2}{3}w \leq g \leq w$, $w \leq g \leq \frac{4}{3}w$ and $\frac{4}{3}w \leq g \leq 1$. It also turns out that in all these cases every inequality of (7) holds except three ones. These are as follows:

$$\varrho \in \bigcap_{\alpha_2} \mathcal{D}_3^{\alpha_2} \implies \begin{array}{c|ccc} \text{case} & (i) & (ii) & (iii) \\ \hline 0 \leq g \leq \frac{2}{3}w & w \leq \frac{3}{11} - \frac{3}{11}g & w \leq 1 - 3g & w \leq \frac{9}{17} + \frac{3}{17}g \\ \frac{2}{3}w \leq g \leq w & w \leq \frac{3}{19} + \frac{3}{19}g & w \leq 1 - 3g & w \leq \frac{9}{17} + \frac{3}{17}g \\ w \leq g \leq \frac{4}{3}w & 3g - \frac{3}{5} \leq w & w \leq 1 - 3g & 3g - \frac{9}{7} \leq w \\ \frac{4}{3}w \leq g \leq 1 & 3g - \frac{3}{5} \leq w & w \leq \frac{3}{11} - \frac{3}{11}g & 3g - \frac{9}{7} \leq w \end{array} \quad (10)$$

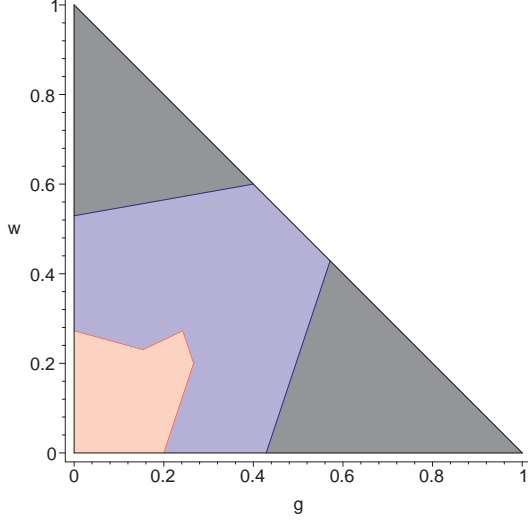


FIG. 2. (Color online) Majorization criterion for state (4) on the g - w -plane. (Red/light grey domain: $\varrho \prec \varrho^1$ and $\varrho \prec \varrho^{23}$, blue/grey domain: $\varrho \prec \varrho^1$ and $\varrho \not\prec \varrho^{23}$, dark grey domains: $\varrho \not\prec \varrho^1$ and $\varrho \not\prec \varrho^{23}$.)

where in columns (i) and (ii) are the first two inequalities of (7) (i.e. $k = 1, 2$) written on $\varrho \prec \varrho^{23}$, and in column (iii) are the first inequalities of (7) written on $\varrho \prec \varrho^1$ in all of the four cases. We can make the inequalities of (10) expressive with the help of Fig. 2. It can be seen that in our case $\varrho \prec \varrho^{23}$ implies $\varrho \prec \varrho^1$, so the bigger subsystem (the trace map on smaller subsystem) gives the stronger condition. (This is not true in general. One can find a permutation invariant three-qubit state where $\varrho \prec \varrho^1$ and $\varrho \prec \varrho^{23}$ can hold independently.)

The rhs. of (9) holds for states of parameters in the red/light grey domain of Fig. 2, so it contains Classes 2.8 and 3. On the other hand, states of parameters in the blue/grey or dark grey domain are in Classes 2.1 or 1, but there can also be such states in the red/light grey domain. Moreover, the union of Classes 2.8 and 3 is a convex set inside the red/light grey domain. In the following we consider some other criteria in order to decrease the area of the red/light grey domain. In this way we can identify more states to be in Classes 2.1 or 1. But before this, we can make an interesting observation here. One can check that for the GHZ-white noise mixture ($w = 0$ line) the majorization criterion $\varrho \prec \varrho^1$ and $\varrho \prec \varrho^{23}$ is necessary and sufficient for full-separability, moreover, the criterion $\varrho \prec \varrho^1$ and $\varrho \not\prec \varrho^{23}$ is necessary and sufficient for Class 2.1, and the criterion $\varrho \not\prec \varrho^1$ and $\varrho \not\prec \varrho^{23}$ is

necessary and sufficient for Class 1. (See Section III for summary of known exact results on the GHZ-white noise mixture.) Hence the condition of genuine three-partite entanglement is the violation of both majorization of (9) for the GHZ-white noise mixture.

B. Entropy criterion

The *Rényi entropy* of a probability distribution p is defined for all $0 \leq \alpha$ as

$$H_\alpha(p) = \frac{1}{1-\alpha} \ln \sum_i p_i^\alpha. \quad (11a)$$

For $\alpha = 0$ this is the logarithm of the number of nonzero p_i s, known as *Hartley entropy*:

$$H_0(i) := \lim_{\alpha \rightarrow 0} H_\alpha(p) = \ln |\{p_i \mid p_i \neq 0\}|. \quad (11b)$$

For $\alpha \rightarrow 1$ it converges to the *Shannon entropy*:

$$H_1(p) := \lim_{\alpha \rightarrow 1} H_\alpha(p) = H(p) = - \sum_i p_i \ln p_i. \quad (11c)$$

For $\alpha \rightarrow \infty$ it converges to the *Chebyshev entropy*:

$$H_\infty(p) := \lim_{\alpha \rightarrow \infty} H_\alpha(p) = - \ln p_{\max}. \quad (11d)$$

The quantum versions of these are defined on density operators and can be calculated as the corresponding entropies of the spectrum. The *quantum-Rényi entropy* for all $0 \leq \alpha$ is

$$S_\alpha(\varrho) = \frac{1}{1-\alpha} \ln \text{Tr}(\varrho^\alpha) = H_\alpha(\text{Spect}(\varrho)). \quad (12a)$$

The *quantum-Hartley entropy* is then:

$$S_0(\varrho) := \lim_{\alpha \rightarrow 0} S_\alpha(\varrho) = \ln \text{rk } \varrho = H_0(\text{Spect}(\varrho)) \quad (12b)$$

the logarithm of the rank of ϱ . For $\alpha \rightarrow 1$ it converges to the *von Neumann entropy*:

$$\begin{aligned} S_1(\varrho) &:= \lim_{\alpha \rightarrow 1} S_\alpha(\varrho) = S(\varrho) = - \text{Tr } \varrho \ln \varrho \\ &= H(\text{Spect}(\varrho)). \end{aligned} \quad (12c)$$

For $\alpha \rightarrow \infty$ it converges to the *quantum-Chebyshev entropy*:

$$\begin{aligned} S_\infty(\varrho) &:= \lim_{\alpha \rightarrow \infty} S_\alpha(\varrho) = - \ln \max \text{Spect}(\varrho) \\ &= H_\infty(\text{Spect}(\varrho)). \end{aligned} \quad (12d)$$

Now we can turn to the *entropy criterion* for two-partite density matrices [23–26]. This is an entropy-based restatement of the statement “for a separable state the whole system is more disordered than its subsystems”:

$$\varrho \text{ separable} \implies S_\alpha(\varrho) \geq S_\alpha(\varrho^A) \quad \text{and} \quad S_\alpha(\varrho) \geq S_\alpha(\varrho^B). \quad (13)$$

The rhs. of (13) can also be true for entangled states, but if it does not hold then the state must be entangled. The entropy criterion follows from the majorization criterion since the Rényi entropies are *Schur concave* functions on the set of probability distributions. (That is if $p \prec q$ then $H_\alpha(p) \geq H_\alpha(q)$.) Therefore *the entropy criterion can not be stronger than the majorization criterion*. In the following we illustrate this with the state ϱ given in Eq. (4) for some particular choice of α .

The rank of ϱ , ϱ^{23} and ϱ^1 can be determined easily due to the simple form of the spectra in Eqs. (6a)-(6c). Hence the entropy criterion for Hartley entropy (12b) can be readily examined. $\text{rk } \varrho = 8$ if and only if $d \neq 0$. The rhs. of (13) holds for these states. It is true for all states that $\text{rk } \varrho^1 = 2$. On the line $w = 1 - g$ ($d = 0$) we have to make distinction between the pure and mixed cases. If $g = 1$ (pure GHZ state) or $w = 1$ (pure W state) then $\text{rk } \varrho^{23} = 2$ and $\text{rk } \varrho = 1$ hence for this case $S_0(\varrho) \not\geq S_0(\varrho^1)$ and $S_0(\varrho) \not\geq S_0(\varrho^{23})$. For the genuine mixtures of GHZ and W states $\text{rk } \varrho^{23} = 3$ and $\text{rk } \varrho = 2$ hence $S_0(\varrho) \geq S_0(\varrho^1)$ but $S_0(\varrho) \not\geq S_0(\varrho^{23})$. So we can establish that *the entropy criterion in the limit $\alpha \rightarrow 0$ (quantum-Hartley entropy) is too weak, it identifies only the GHZ-W mixture to be entangled*.

Consider now the entropy criterion in the $\alpha \rightarrow \infty$ limit. This can easily be done because the inequalities of the rhs. of (13) are the same as the ones in the (i)th and (iii)th column of (10), which are written on the maximal eigenvalues. Hence in this case we have fewer restrictions, and one can see in Fig. 3 that the rhs. of (13) holds for more states than the rhs. of (9) in the case of the majorization criterion. Hence *the entropy criterion in the $\alpha \rightarrow \infty$ limit (quantum-Chebyshev entropy) identifies a little bit fewer state to be entangled than the majorization criterion*.

Increasing α from 0 to ∞ one can see in Fig. 4 how the borderlines of the domains of the entropy criterion shrink to the ones in Fig. 3. It is not true in general that if $H_\alpha(p) \leq H_\alpha(q)$ and $\alpha \geq \beta$ then $H_\beta(p) \leq H_\beta(q)$. For these particular spectra it seems that the domains of smaller α s would contain the domains of larger α s, but for the large values of α one can see that this is not true. However, no line can cross the border of the domain of majorization criterion, since the entropy criterion can not be stronger than the majorization criterion.

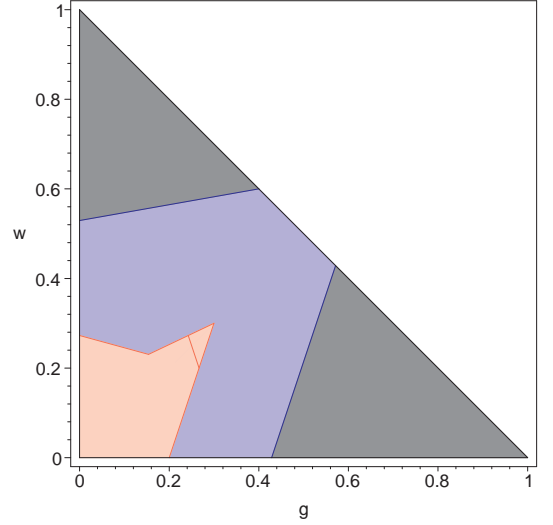


FIG. 3. (Color online) Entropy criterion in the limit $\alpha \rightarrow \infty$ for state (4) on the g - w -plane. (Red/light grey domain: $S_\infty(\varrho) \geq S_\infty(\varrho^1)$ and $S_\infty(\varrho) \geq S_\infty(\varrho^{23})$, blue/grey domain: $S_\infty(\varrho) \geq S_\infty(\varrho^1)$ and $S_\infty(\varrho) \not\geq S_\infty(\varrho^{23})$, dark grey domains: $S_\infty(\varrho) \not\geq S_\infty(\varrho^1)$ and $S_\infty(\varrho) \not\geq S_\infty(\varrho^{23})$.)

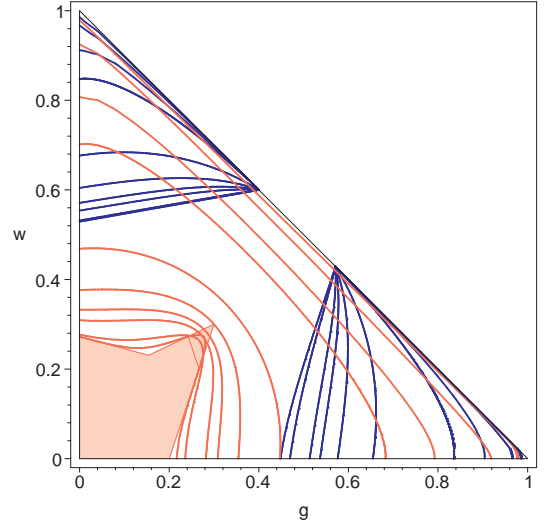


FIG. 4. (Color online) Entropy criterion for $\alpha = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, 2, 3, 4, 5, 10, 20$ for state (4) on the g - w -plane. (Red/grey curves: the border of the domain $S_\alpha(\varrho) \geq S_\alpha(\varrho^1)$ and $S_\alpha(\varrho) \geq S_\alpha(\varrho^{23})$, blue/black curves: the border of the domain $S_\alpha(\varrho) \geq S_\alpha(\varrho^1)$ and $S_\alpha(\varrho) \not\geq S_\alpha(\varrho^{23})$. Red/light grey domain: copied from Fig. 3 of the $\alpha \rightarrow \infty$ case.)

C. Partial transposition criterion

If a two-partite state is separable then the partial transposition on subsystem A acts on the ϱ_i^A s of the decomposition given in Eq. (1). The transposition does not change the eigenvalues of a self-adjoint matrix, so $(\varrho_i^A)^T$ s are also density matrices (i.e. self-adjoint matrices of trace one). Hence the partial transpose of a separable density matrix is also a density matrix. (Its eigenvalues

are not the same in general as the ones of the original matrix, but they are also *nonnegative* and sum up to one.) The reverse is not true unless the system is of qubit-qubit or qubit-qutrit [8], so our 2×4 system is the smallest one for which this implication is only one-way. In general we get the *partial transposition criterion* of Peres [1]:

$$\rho \text{ separable} \implies \rho^{TA} \geq 0. \quad (14)$$

It is clear that no matter which subsystem is transposed. (The partial transposition criterion is the consequence of the *positive maps criterion*: [8])

$$\rho \text{ separable} \iff (\Phi \otimes \mathbb{I})\rho \geq 0 \text{ for all positive maps } \Phi. \quad (15)$$

This is a necessary and sufficient criterion, but we can not check it for all Φ . But we can consider a particular class of positive maps to obtain necessary but not sufficient criteria. For example for $\Phi(\omega) = \omega^T$ we get back the partial transposition criterion.)

Let us apply the partial transposition criterion to the state ρ of Eq. (4). The spectrum of ρ^{T_1} can easily be calculated due to its block-structure. (See in Eq. (A2) of the Appendix.)

$$\begin{aligned} \text{Spect}(\rho^{T_1}) = & \\ & \{ \tilde{d} + \tilde{w}/2 \pm \sqrt{4\tilde{g}^2 + \tilde{w}^2}/2 \\ & = (3 - 3g + w \pm 4\sqrt{9g^2 + w^2})/24, \\ & \tilde{d} + \tilde{g}/2 \pm \sqrt{\tilde{g}^2 + 8\tilde{w}^2}/2 \\ & = (3 + 3g - 3w \pm 2\sqrt{9g^2 + 32w^2})/24, \\ & \tilde{d} + \tilde{g} = (3 + 9g - 3w)/24, \\ & \tilde{d} + 2\tilde{w} = (3 - 3g + 13w)/24, \\ & \tilde{d} = (3 - 3g - 3w)/24 \quad 2 \text{ times} \}. \end{aligned} \quad (16)$$

Only the minus-version of the first two eigenvalues can be less than zero hence we get two inequalities for the positivity of ρ^{T_1} :

$$\begin{aligned} \rho \in \bigcap_{\alpha_2} \mathcal{D}_3^{\alpha_2} & \implies \\ \begin{cases} 0 \leq \tilde{d}^2 + \tilde{d}\tilde{w} - \tilde{g}^2 \\ 0 \leq -135g^2 - 15w^2 - 6gw - 18g + 6w + 9, \end{cases} & (17a) \\ \begin{cases} 0 \leq \tilde{d}^2 + \tilde{d}\tilde{g} - 2\tilde{w}^2 \\ 0 \leq -27g^2 - 119w^2 - 18gw + 18g - 18w + 9. \end{cases} & (17b) \end{aligned}$$

Each inequality of these holds inside an ellipse. These ellipses intersect nontrivially and in the intersection the rhs. of (14) holds. (Red/light grey curves in Fig. 5.) $g = 1/5$ and $w = (24\sqrt{2} - 9)/119 = 0.209589\dots$ are the bounds for the union of Class 2.8 and 3 for the GHZ-white

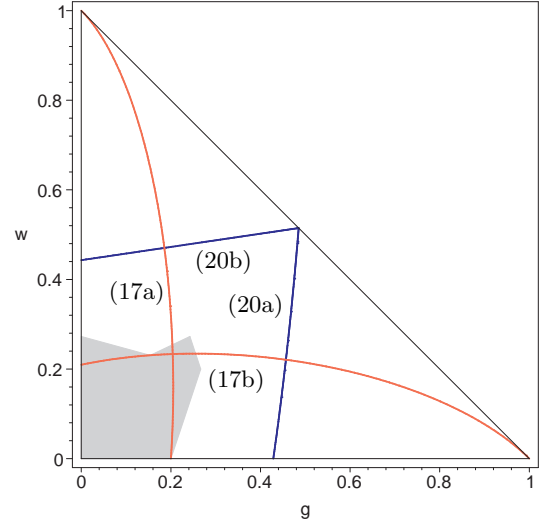


FIG. 5. (Color online) Partial transposition and reduction criteria for state (4) on the g - w -plane. (Eqs. (17a) and (17b) of partial transposition criterion hold inside the intersection of the red/grey ellipses. Blue/black curves: the borders of domains inside the additional Eqs. (20a) and (20b) of reduction criterion hold. Red/light grey domain: copied from Fig. 2 of majorization criterion.)

noise ($w = 0$) and the W-white noise ($g = 0$) mixtures respectively.

The partial transposition criterion states then that if a state is in Classes 2.8 or 3 then its parameters are inside the intersection of the ellipses, but there can also be states of Classes 2.1 or 1 in there. On the other hand the states must be in Classes 2.1 or 1 for parameters outside. *The inequalities of (17a) and (17b) are strong in detection of GHZ and W state respectively.* In Fig. 5 we have also plotted the corresponding domain of the majorization criterion. (One can check that the only intersection-points of the borderlines of the corresponding domains of the two criteria are $(g = 2/13, w = 3/13)$ and $(g = 1/5, w = 0)$. This criterion is also a necessary and sufficient one for the full separability of the $w = 0$ GHZ-white noise mixture.) It can be seen that *the partial transposition criterion gives stronger condition than the majorization criterion*, it identifies more state to be in Classes 2.1 or 1. Hence the majorization criterion can not identify entangled states of positive partial transpose (PPTES) on the simplex defined in Eq. (4).

The PPTESs are exotic entangled states. They are *bound entangled (undistillable)* states, [27] that is entangled states from which no entanglement can be distilled at all. (The entanglement distillation [28, 29] is a family of methods which allow one to extract locally maximally entangled pure states out of a given state or its copies.) It is usually hard to check that a state of positive partial transpose is not separable, there are few explicit examples of PPTES in the literature (see a list of references in Section 1.2.4 of Ref. [29]). All the states in Class 2.8 are PPTESs.

D. Reduction criterion

The next one of the examined criteria is the *reduction criterion*. [30, 31] It states that

$$\varrho \text{ separable} \implies \varrho^A \otimes \mathbb{I}^B - \varrho \geq 0 \quad \text{and} \quad \mathbb{I}^A \otimes \varrho^B - \varrho \geq 0. \quad (18)$$

This is the consequence of the positive maps criterion given in (15) for the particular positive map $\Phi(\omega) = \text{Tr}(\omega)\mathbb{I} - \omega$. The importance of this criterion is that its violation is sufficient criterion of *distillability*. [30] It is known [30] that *the reduction criterion can not be stronger than the partial transposition criterion and they are equivalent for qubit-qudit systems*. Since our state ϱ defined in Eq. (4) is the permutation invariant one of three qubits considered as a 2×4 qubit-qudit system, the equivalence of these two criteria means that some kind of pure state entanglement between 1 and 23 can be distilled out from every state of non positive partial transpose. In other words *in the simplex defined by Eq. (4) there are no bound entangled 2×4 states of non positive partial transpose*.

We can illustrate the equivalence of the partial transposition and reduction criteria. To do this we have to examine the positivity of the matrices $\mathbb{I}^1 \otimes \varrho^{23} - \varrho$ and $\varrho^1 \otimes \mathbb{I}^{23} - \varrho$. (See in Eqs. (A5) and (A6) of the Appendix.) Since $\text{Tr}(\omega)\mathbb{I} - \omega = (\sigma_2 \omega \sigma_2)^T$ for 2×2 matrices (with the Pauli matrix $\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$) it turns out that

$$\begin{aligned} \text{Spect}(\mathbb{I}^1 \otimes \varrho^{23} - \varrho) &= \text{Spect}(\varrho^{T_1}), \\ \text{Spect}(\varrho^1 \otimes \mathbb{I}^{23} - \varrho) &= \\ &\{3\tilde{d} + 3\tilde{w}/2 \pm \sqrt{4\tilde{g}^2 + \tilde{w}^2}/2 \\ &= (9 - 9g + 3w \pm 4\sqrt{9g^2 + w^2})/24, \\ 3\tilde{d} + \tilde{g} \pm \sqrt{2}\tilde{w} &= (9 + 3g - 9w \pm 8\sqrt{2}w)/24, \\ 3\tilde{d} + \tilde{g} + 2\tilde{w} &= (9 + 3g + 7w)/24 \quad 2 \text{ times}, \\ 3\tilde{d} + \tilde{g} + \tilde{w} &= (9 + 3g - w)/24 \quad 2 \text{ times} \}. \end{aligned} \quad (19a)$$

For $\mathbb{I}^1 \otimes \varrho^{23} - \varrho \geq 0$ we have the same conditions as in Eqs. (17a)-(17b) of the partial transposition criterion. The additional inequalities arise from the minus-version of the first two eigenvalues of $\varrho^1 \otimes \mathbb{I}^{23} - \varrho$:

$$\begin{aligned} \varrho \in \bigcap_{\alpha_2} \mathcal{D}_3^{\alpha_2} &\implies \\ \begin{cases} 0 \leq 9\tilde{d}^2 - \tilde{g}^2 + 9\tilde{d}\tilde{w} + 2\tilde{w}^2 \\ 0 \leq -63g^2 - 7w^2 - 54gw - 162g + 54w + 81, \end{cases} \end{aligned} \quad (20a)$$

$$\begin{cases} 0 \leq 3\tilde{d} + \tilde{g} - \sqrt{2}\tilde{w} \\ 0 \leq 3g - (9 + 8\sqrt{2})w + 9. \end{cases} \quad (20b)$$

The first one of them is true outside a hyperbola, the second one is true under a line. (Blue/black curves in Fig. 5.)

It can be seen that the last two inequalities (20a)-(20b) do not restrict the ones in Eqs. (17a)-(17b), as it has to be, and because of (19a) *the reduction criterion and the partial transposition criterion hold for the same states of the GHZ-W-white noise mixture*. Here we get the stronger condition for the map $\Phi(\omega) = \text{Tr}(\omega)\mathbb{I} - \omega$ acting on the smaller subsystem. We can also observe that the inequalities of (20a) and (20b) are good in detection of GHZ and W state respectively, but not so good as the ones of partial transposition criterion. However, one can check that on the $w = 0$ GHZ-white noise mixture the reduction criterion $\mathbb{I}^1 \otimes \varrho^{23} - \varrho \geq 0$ and $\varrho^1 \otimes \mathbb{I}^{23} - \varrho \geq 0$ is necessary and sufficient for full-separability, the criterion $\mathbb{I}^1 \otimes \varrho^{23} - \varrho \not\geq 0$ and $\varrho^1 \otimes \mathbb{I}^{23} - \varrho \geq 0$ is necessary and sufficient for Class 2.1, and the criterion $\mathbb{I}^1 \otimes \varrho^{23} - \varrho \not\geq 0$ and $\varrho^1 \otimes \mathbb{I}^{23} - \varrho \not\geq 0$ is necessary and sufficient for Class 1 in the same fashion as in the majorization criterion of Section IV A.

E. Reshuffling criterion

The *reshuffling criterion* is independent of the partial transposition criterion, so it can detect entangled states of positive partial transpose. It states that:

$$\varrho \text{ separable} \implies \|R(\varrho)\|_{\text{Tr}} \leq 1, \quad (21)$$

where the *trace-norm* is $\|A\|_{\text{Tr}} = \text{Tr} \sqrt{A^\dagger A}$, and the *reshuffling map* R is defined on matrix elements as $[R(\varrho)]_{ii',jj'} = \varrho_{ij,i'j'}$.

The four nonzero singular values of the 4×16 reshuffled density matrix, (see in Eq. (A7) of the Appendix) i.e. the square root of the nonnegative eigenvalues of $R(\varrho)R(\varrho)^\dagger$ are:

$$\begin{aligned} \text{Spect} \sqrt{R(\varrho)R(\varrho)^\dagger} \\ = \left\{ \sqrt{p_1 \pm 2\sqrt{p_2}}/2, \sqrt{\tilde{g}^2 + 2\tilde{w}^2}, \sqrt{\tilde{g}^2 + 2\tilde{w}^2} \right\}, \end{aligned} \quad (22)$$

where

$$\begin{aligned} p_1 &= 16\tilde{d}^2 + 4\tilde{g}^2 + 10\tilde{w}^2 + 8\tilde{d}\tilde{g} + 12\tilde{d}\tilde{w}, \\ p_2 &= 64\tilde{d}^4 + 9\tilde{w}^4 + 64\tilde{d}^3\tilde{g} + 96\tilde{d}^3\tilde{w} + 12\tilde{d}\tilde{w}^3 \\ &\quad + 16\tilde{d}^2\tilde{g}^2 + 40\tilde{d}^2\tilde{w}^2 + 4\tilde{g}^2\tilde{w}^2 \\ &\quad + 80\tilde{d}^2\tilde{g}\tilde{w} + 16\tilde{d}\tilde{g}^2\tilde{w} + 24\tilde{d}\tilde{g}\tilde{w}^2. \end{aligned}$$

The sum of them is less or equal than 1 inside a curve of high degree which can be seen in Fig. 6 (red/grey curve). States of Classes 2.8 and 3 must be inside this curve, states outside this curve must belong to Classes 2.1 or 1, but one can see that *this criterion does not restrict the partial transposition criterion*, it does not detect PPTESs in the GHZ-W-white noise mixture of Eq. (4).

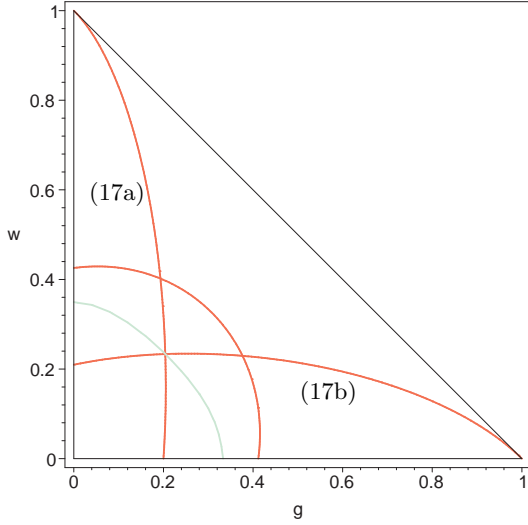


FIG. 6. (Color online) Reshuffling criteria for state (4) on the g - w -plane. (Red/grey curve: reshuffling criterion for 2×4 system as in Section IV E, green/light grey curve: reshuffling criterion for $2 \times 2 \times 2$ system as in Section V A. We have also copied the borderlines of the domains in which (17a) and (17b) of the partial transposition criterion hold from Fig. 5. The inequalities hold on the side of the curves containing the origin.)

V. THREE-PARTITE SEPARABILITY CRITERIA

In this section we consider our system given in Eq. (4) as a proper $2 \times 2 \times 2$ three-qubit one and investigate some general 3-qubit k -separability criteria.

A. Permutation criterion

First consider the *permutation criterion* in general [32]. Note that the reshuffling and the partial transpose of a density matrix are nothing else than the permutation of the local matrix indices. Moreover, since the trace norm is the sum of the absolute values of the eigenvalues for hermitian matrices and the trace is invariant under partial transposition it turns out that $\varrho^{T_1} \geq 0$ if and only if $\|\varrho^{T_1}\|_{\text{Tr}} = 1$. So the partial transposition criterion (14) and the reshuffling criterion (21) can be formulated in the same fashion. Moreover, this can be done for N subsystems in a general way. [32] Let $\pi \in S_{2N}$ a permutation of the $2N$ matrix indices and let Λ_π the map realizing this index permutation: if $\varrho = \sum \varrho_{i_1 i_2 \dots i_N, i_{N+1} i_{N+2} \dots i_{2N}} |i_1 i_2 \dots i_N\rangle \langle i_{N+1} i_{N+2} \dots i_{2N}|$ then $[\Lambda_\pi(\varrho)]_{i_{\pi(1)} i_{\pi(2)} \dots i_{\pi(N)}, i_{\pi(N+1)} i_{\pi(N+2)} \dots i_{\pi(2N)}} = \varrho_{i_1 i_2 \dots i_N, i_{N+1} i_{N+2} \dots i_{2N}}$. Now the *permutation criterion* states that

$$\varrho \text{ fully separable} \implies \|\Lambda_\pi(\varrho)\|_{\text{Tr}} \leq 1, \quad \forall \pi \in S_{2N}. \quad (23)$$

The permutation criterion gives $(2N)!$ criteria but not all of them are inequivalent. It is well known [32] that for two subsystems, every criteria given by the permutation criterion turn out to be equivalent either the partial transposition criterion or the reshuffling criterion. In Ref. [33] Clarisse has shown that there are only six inequivalent criteria in the case of three subsystems: three one-partite-transpositions and three two-partite-reshufflings. For our *permutation-invariant* three-qubit system all the one-partite-transpositions give the same condition which we have already investigated in Section IV C. On the other hand, all the two-partite-reshufflings give another condition which is a new one.

So let $R' = \Lambda_\pi$ the map implementing the reshuffling of the 2 and 3 subsystems: $[R'(\varrho)]_{i j j', i' k k'} = \varrho_{i j k, i' j' k'}$. (See in Eq. (A8) of the Appendix.) With this we have to calculate the eigenvalues of the matrix $R'(\varrho)R'(\varrho)^\dagger$ for the two-parameter state ϱ given in Eq. (4). This 8×8 matrix can be transformed (by simultaneous row-column permutation) into blockdiagonal form consisting three blocks of the shape 3×3 , 3×3 and 2×2 . However, the forms of the g, w -depending eigenvalues of the 3×3 blocks are still too complicated, so we only plot the border of the domain in which the criterion (23) holds. (See green/light grey curve in Fig. 6.)

The condition $\|R'(\varrho)\|_{\text{Tr}} \leq 1$ holds inside the green/light grey curve in Fig. 6. This figure suggests that this reshuffling does not give stricter condition for full separability than the partial transposition criterion, hence it can not identify PPTESs. However, we can not be sure in this due to the difficult computation of $\|R'(\varrho)\|_{\text{Tr}}$. Fully separable states must be enclosed by the curves belonging to (17a)-(17b) of partial transposition criterion, states outside this domain must belong to Classes 2.8, 2.1 or 1. However, in Section IV C the partial transposition criterion has yielded condition for Classes 2.8 and 3, so we can conclude that states outside this domain must belong to Classes 2.1 or 1, the Class 2.8 is completely restricted into this domain.

B. Criteria on spin-observables

In Ref. [10] Seevinck and Uffink introduced a systematic way to obtain necessary criteria of separability for all the separability-classes of an N -qubit system. Their criteria generalize some previously known criteria, (see Refs. in Ref. [10]) such as Laskowski-Żukowski criterion (necessary for k -separability), Mermin-type separability inequalities (necessary for k -separability), Fidelity-criterion (necessary for 2-separability) and Dür-Cirac depolarization criterion (necessary for α_k -separability). We consider the three-qubit case and get criteria for Class 2.1, Class 2.8 and Class 3 given in Section II.

The method of Seevinck and Uffink deals with three orthogonal *spin-observables* on each subsystem: $(X^{(1)}, Y^{(1)}, Z^{(1)})$. Here the superscript (1) denotes that these are single-qubit operators. Let $I^{(1)}$ denote the

2×2 identity matrix. From the $(X^{(1)}, Y^{(1)}, Z^{(1)}, I^{(1)})$ one-qubit observables acting on the 2 and 3 subsystem one can form two sets of two-qubit observables: $(X_x^{(2)}, Y_x^{(2)}, Z_x^{(2)}, I_x^{(2)})$. Here the superscript (2) denotes that these are two-qubit operators and $x = 0, 1$ refers to the two sets:

$$\begin{aligned} X_0^{(2)} &= \frac{1}{2} (X^{(1)} \otimes X^{(1)} - Y^{(1)} \otimes Y^{(1)}), \\ X_1^{(2)} &= \frac{1}{2} (X^{(1)} \otimes X^{(1)} + Y^{(1)} \otimes Y^{(1)}), \\ Y_0^{(2)} &= \frac{1}{2} (Y^{(1)} \otimes X^{(1)} + X^{(1)} \otimes Y^{(1)}), \\ Y_1^{(2)} &= \frac{1}{2} (Y^{(1)} \otimes X^{(1)} - X^{(1)} \otimes Y^{(1)}), \\ Z_0^{(2)} &= \frac{1}{2} (Z^{(1)} \otimes I^{(1)} + I^{(1)} \otimes Z^{(1)}), \\ Z_1^{(2)} &= \frac{1}{2} (Z^{(1)} \otimes I^{(1)} - I^{(1)} \otimes Z^{(1)}), \\ I_0^{(2)} &= \frac{1}{2} (I^{(1)} \otimes I^{(1)} + Z^{(1)} \otimes Z^{(1)}), \\ I_1^{(2)} &= \frac{1}{2} (I^{(1)} \otimes I^{(1)} - Z^{(1)} \otimes Z^{(1)}). \end{aligned} \quad (24)$$

(Note that $I_x^{(2)}$ s are *not* identity operators.) From this two-qubit observables and the one-qubit ones acting on the 1 subsystem one can form four sets of three-qubit observables acting on the full system: $(X_x^{(3)}, Y_x^{(3)}, Z_x^{(3)}, I_x^{(3)})$. Here the superscript (3) denotes that these are three-qubit operators and $x = 0, 1, 2, 3$ refers to the four sets:

$$\begin{aligned} X_y^{(3)} &= \frac{1}{2} (X^{(1)} \otimes X_{y/2}^{(2)} - Y^{(1)} \otimes Y_{y/2}^{(2)}), \\ X_{y+1}^{(3)} &= \frac{1}{2} (X^{(1)} \otimes X_{y/2}^{(2)} + Y^{(1)} \otimes Y_{y/2}^{(2)}), \\ Y_y^{(3)} &= \frac{1}{2} (Y^{(1)} \otimes X_{y/2}^{(2)} + X^{(1)} \otimes Y_{y/2}^{(2)}), \\ Y_{y+1}^{(3)} &= \frac{1}{2} (Y^{(1)} \otimes X_{y/2}^{(2)} - X^{(1)} \otimes Y_{y/2}^{(2)}), \\ Z_y^{(3)} &= \frac{1}{2} (Z^{(1)} \otimes I_{y/2}^{(2)} + I^{(1)} \otimes Z_{y/2}^{(2)}), \\ Z_{y+1}^{(3)} &= \frac{1}{2} (Z^{(1)} \otimes I_{y/2}^{(2)} - I^{(1)} \otimes Z_{y/2}^{(2)}), \\ I_y^{(3)} &= \frac{1}{2} (I^{(1)} \otimes I_{y/2}^{(2)} + Z^{(1)} \otimes Z_{y/2}^{(2)}), \\ I_{y+1}^{(3)} &= \frac{1}{2} (I^{(1)} \otimes I_{y/2}^{(2)} - Z^{(1)} \otimes Z_{y/2}^{(2)}), \end{aligned} \quad (25)$$

for $y = 0, 2$. (Note that $I_x^{(3)}$ s are *not* identity operators.)

Now for particular α_2 investigating some relations among the expectation-values of these operators with respect to the state ϱ one can get some nontrivial inequalities valid for all $\varrho \in \mathcal{D}_3^{\alpha_2}$. From these one can form inequalities valid for a given separability class of Section II. Here we recall [10] these criteria for the classes we

need to deal with:

$$\varrho \in \mathcal{D}_3^{2\text{-sep}} \implies \sqrt{\langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2} \leq \sum_{y \neq x} \sqrt{\langle I_y^{(3)} \rangle^2 - \langle Z_y^{(3)} \rangle^2} \quad (26)$$

for $x = 0, 1, 2, 3$,

$$\begin{aligned} \varrho \in \bigcap_{\alpha_2} \mathcal{D}_3^{\alpha_2} \implies \max_x \{ \langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2 \} \\ \leq \min_x \{ \langle I_x^{(3)} \rangle^2 - \langle Z_x^{(3)} \rangle^2 \} \leq 1/4 \end{aligned} \quad (27)$$

and

$$\begin{aligned} \varrho \in \mathcal{D}_3^{3\text{-sep}} \implies \max_x \{ \langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2 \} \\ \leq \min_x \{ \langle I_x^{(3)} \rangle^2 - \langle Z_x^{(3)} \rangle^2 \} \leq 1/16. \end{aligned} \quad (28)$$

One has to do some optimisation of the local spin observables $(X^{(1)}, Y^{(1)}, Z^{(1)})$ to get violation of the respective inequality for a given state.

In the following we will consider some special measurement-settings when the observables $(X^{(1)}, Y^{(1)}, Z^{(1)})$ are the same for each subsystem. Writing out explicitly $(X_x^{(3)}, Y_x^{(3)}, Z_x^{(3)}, I_x^{(3)})$, one can see that *for a permutation-invariant state* the squares of the expectation values are the same for $x = 1, 2, 3$, i.e.: $\langle X_1^{(3)} \rangle^2 = \langle X_2^{(3)} \rangle^2 = \langle X_3^{(3)} \rangle^2$, and the same for $Y_x^{(3)}$ s, $Z_x^{(3)}$ s and $I_x^{(3)}$ s. Hence we have to consider merely the $x = 0, 1$ indices.

First consider the special choice when $(X^{(1)}, Y^{(1)}, Z^{(1)}) = (\sigma_x, \sigma_y, \sigma_z)$ for each subsystem. (Here σ_i s denote the usual Pauli matrices.) We refer to this as Setting I. The inequalities (26)-(28) can be written as relatively simple expressions in the matrix elements [10]:

$$\begin{aligned} \varrho \in \mathcal{D}_3^{2\text{-sep}} \implies \begin{aligned} |\varrho_{07}| &\leq \sqrt{\varrho_{66}\varrho_{11}} + \sqrt{\varrho_{55}\varrho_{22}} + \sqrt{\varrho_{33}\varrho_{44}}, \\ |\varrho_{61}| &\leq \sqrt{\varrho_{00}\varrho_{77}} + \sqrt{\varrho_{55}\varrho_{22}} + \sqrt{\varrho_{33}\varrho_{44}}, \\ |\varrho_{52}| &\leq \sqrt{\varrho_{66}\varrho_{11}} + \sqrt{\varrho_{00}\varrho_{77}} + \sqrt{\varrho_{33}\varrho_{44}}, \\ |\varrho_{34}| &\leq \sqrt{\varrho_{66}\varrho_{11}} + \sqrt{\varrho_{55}\varrho_{22}} + \sqrt{\varrho_{00}\varrho_{77}}, \end{aligned} \end{aligned} \quad (29)$$

$$\begin{aligned} \varrho \in \bigcap_{\alpha_2} \mathcal{D}_3^{\alpha_2} \implies \max \{ |\varrho_{07}|^2, |\varrho_{61}|^2, |\varrho_{52}|^2, |\varrho_{34}|^2 \} \\ \leq \min \{ \varrho_{00}\varrho_{77}, \varrho_{66}\varrho_{11}, \varrho_{55}\varrho_{22}, \varrho_{33}\varrho_{44} \} \leq 1/16 \end{aligned} \quad (30)$$

and

$$\begin{aligned} \varrho \in \mathcal{D}_3^{3\text{-sep}} \implies \max \{ |\varrho_{07}|^2, |\varrho_{61}|^2, |\varrho_{52}|^2, |\varrho_{34}|^2 \} \\ \leq \min \{ \varrho_{00}\varrho_{77}, \varrho_{66}\varrho_{11}, \varrho_{55}\varrho_{22}, \varrho_{33}\varrho_{44} \} \leq 1/64. \end{aligned} \quad (31)$$

(Here the matrix indices run from 0 to 7 so as to be equal to the binary indices we use later.) Let us consider another two special measurement settings: Setting II: $(X^{(1)}, Y^{(1)}, Z^{(1)}) = (\sigma_y, \sigma_z, \sigma_x)$ for each subsystem, Setting III: $(X^{(1)}, Y^{(1)}, Z^{(1)}) = (\sigma_z, \sigma_x, \sigma_y)$ for each subsystem. The inequalities of (26)-(28) written for these two settings are much more complicated expressions in symbolic matrix elements than the ones in (29)-(31). But for the state ϱ given in Eq. (4) it is not too difficult to write out these inequalities explicitly. It turns out that for each of these three settings the $x = 1$ inequality of (26), the second inequality of (27) and the second inequality of (28) hold for all the parameter values of the simplex. Because of this, the criteria hold for Class 3 are not stricter than the ones for the union of Class 2.8 and Class 3. The remaining inequalities for the three measurement settings are as follows:

$$\varrho \in \mathcal{D}_3^{2\text{-sep}} \implies \text{I.} \begin{cases} \tilde{g} \leq 3\sqrt{\tilde{d}(\tilde{d} + \tilde{w})} \\ 0 \leq -7g^2 - 6gw - 15w^2 - 18g + 6w + 9, \end{cases} \quad (32a)$$

$$\text{II.} \begin{cases} 3\tilde{w} \leq \sqrt{(8\tilde{d} + \tilde{w})(8\tilde{d} + 4\tilde{g} + \tilde{w})} \\ 0 \leq -9g^2 - 5w^2 - 12w + 9, \end{cases} \quad (32b)$$

$$\text{III.} \begin{cases} \sqrt{4\tilde{g}^2 + 81\tilde{w}^2} \leq 3(8\tilde{d} + 2\tilde{g} + \tilde{w}) \\ 0 \leq -g^2 - 5w^2 - 12w + 9, \end{cases} \quad (32c)$$

and

$$\varrho \in \bigcap_{\alpha_2} \mathcal{D}_3^{\alpha_2} \implies \text{I.} \begin{cases} \tilde{g}^2 \leq \tilde{d}(\tilde{d} + \tilde{w}) \\ 0 \leq -45g^2 - 2gw - 5w^2 - 6g + 2w + 3, \end{cases} \quad (33a)$$

$$\text{II.} \begin{cases} 81\tilde{w}^2 \leq (8\tilde{d} + \tilde{w})(8\tilde{d} + 4\tilde{g} + \tilde{w}) \\ 0 \leq -9g^2 - 77w^2 - 12w + 9, \end{cases} \quad (33b)$$

$$\text{III.} \begin{cases} 4\tilde{g}^2 + 81\tilde{w}^2 \leq (8\tilde{d} + 2\tilde{g} + \tilde{w})^2 \\ 0 \leq -9g^2 - 77w^2 - 12w + 9, \end{cases} \quad (33c)$$

Clearly, the inequality of (32c) is weaker than the one of (32b), the inequality of (33c) is the same as the one of (33b). Moreover, the inequality of (33a) is the same as the one of (17a) of partial transposition criterion, but the inequality of (33b) is strictly weaker than the other one of partial transposition criterion. *So these settings does not give stricter conditions for Classes 2.8 and 3 than partial transposition criterion*, however, we get criteria for biseparability for the first time. In Fig. 7 we show the borderlines of the domains of the criteria belonging to Settings I. and II. These inequalities restrict Classes 2.1, 2.8 and 3 to be inside the domain enclosed by the blue/black curves and Classes 2.8 and 3 to be inside the domain enclosed by the red/grey curves. *We can conclude that Settings I. and II. are strong in detection of GHZ and W state respectively.* One can check that for

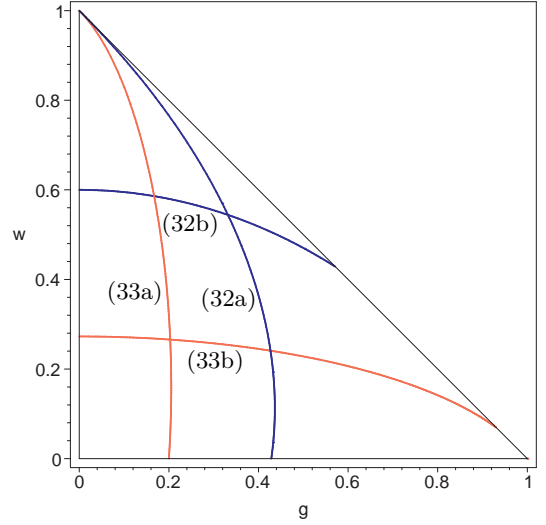


FIG. 7. (Color online) Criteria on spin-observables for state (4) on the g - w -plane. (Red/grey curves: the border of domains inside Eqs. (32a) and (32b) hold, blue/black curves: the border of domains inside Eqs. (33a) and (33b) hold. The inequalities hold on the side of the curves containing the origin.)

the $w = 0$ GHZ-white noise mixture the inequalities of (32a) and (33a) of Setting I. hold if and only if the state is fully separable, (32a) is violated but (33a) holds if and only if the state is in Class 2.1 and both of them are violated if and only if the state is fully entangled. For the $g = 0$ W-white noise mixture if $3/11 < w$ then ϱ is in Class 2.1 or Class 1, and if $3/5 < w$ then ϱ is fully entangled.

However, there are infinitely many criteria depending on the measurement settings and we do not have a method to find a set of settings leading to the strictest criterion. We have tried some other randomly chosen settings which can be used to reduce the area where the criteria hold. We could not find settings that give stronger criteria on the $w = 0$ or $g = 0$ axes of the simplex than Settings I. and II. respectively. We have found settings that excludes states from the corresponding classes, but these states are far from these axes, and we have not found settings which give stronger condition for Classes 2.8 and 3 than the partial transposition criterion. We have found settings by which the condition for biseparability can be strengthened, but these conditions are just a little bit stronger far from the axes than the ones in Section V D.

C. Criteria on matrix elements

In a recent paper [34], Gabriel *et. al.* have given criterion for k -separability, based on their previously derived framework for the detection of biseparability [35]. It turns out that for the noisy GHZ-W mixture given in Eq. (4) these criteria give the same results as the ones of

Seevinck and Uffink, given in the previous section, but these criteria have the advantage that they can be used *in the same form* not only for qubits, but for subsystems of arbitrary, even different dimensions. As our knowledge, these are the only such criteria of k -separability.

Consider some permutation operators acting on $\mathcal{H} \otimes \mathcal{H}$, i.e. on the two copies of the N -partite Hilbert space $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2 \otimes \dots \otimes \mathcal{H}^N$. Let P_a the operators which swap the a th subsystems of the two copies: $P_a|i_1 i_2 \dots i_N\rangle \otimes |j_1 j_2 \dots j_N\rangle = |i_1 i_2 \dots i_{a-1} j_a i_{a+1} \dots i_N\rangle \otimes |j_1 j_2 \dots j_{a-1} i_a j_{a+1} \dots j_N\rangle$ where $\{|i_a\rangle\}$ is basis in \mathcal{H}^a . Now for a composite subsystem $\mathcal{H}^S = \otimes_{a \in S} \mathcal{H}^a$ let $P_S = \prod_{a \in S} P_a$. The key fact is that if the state of a set of subsystems can be separated from the rest of the state then the corresponding P_S leaves the two copies of the state invariant: $P_S^\dagger \varrho^{\otimes 2} P_S = \varrho^{\otimes 2}$. With this and convexity arguments, one can get the following criteria for k -separability [34]:

$$\varrho \in \mathcal{D}_N^{k\text{-sep}} \implies \sqrt{\langle \Phi | \varrho^{\otimes 2} P_{\text{tot}} | \Phi \rangle} \leq \sum_i \left(\prod_{n=1}^k \langle \Phi | P_{S_n^{(i)}}^\dagger \varrho^{\otimes 2} P_{S_n^{(i)}} | \Phi \rangle \right)^{\frac{1}{2k}}, \quad (34)$$

where $|\Phi\rangle \in \mathcal{H} \otimes \mathcal{H}$ is a *fully separable* vector, and the total swap operator is $P_{\text{tot}} = \prod_{a=1}^N P_a$. Here i runs over all possible k -partite splits $\alpha_k^{(i)} = (S_1^{(i)}, S_2^{(i)}, \dots, S_k^{(i)})$.

The inequality in (34) is written on the matrix elements of ϱ determined by the separable detection-vector $|\Phi\rangle$. For a given state, optimisation on $|\Phi\rangle$ is needed to achieve the violation of (34).

To apply these criteria to the noisy GHZ-W mixture given in Eq. (4) we have to choose a suitable detection-vector $|\Phi\rangle$. It turns out that $|\Phi_{\text{GHZ}}\rangle = |000111\rangle$ and $|\Phi_{\text{W}}\rangle = H^{\otimes 6}|\Phi_{\text{GHZ}}\rangle$ are good choices for states in the vicinity of GHZ and W states respectively, as observed in Ref. [34]. (Here $H = 1/\sqrt{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is matrix of the usual Hadamard/discrete Fourier transformation for qubits.) With these two vectors we get the same criteria for 2-separability as the ones in (32a) and (32b) respectively, and for 3-separability as the ones in (33a) and (33b) respectively. However, (33a) and (33b) obtained by the criteria on spin observables are criteria not only for Class 3, but for the union of Classes 2.8 and 3, so in this sense the criteria on spin observables are a bit stronger.

We can not be sure that the detection-vectors above give the strongest conditions at least for the noisy GHZ and noisy W states. However, it is an interesting observation that the Hadamard transformation relates not only the two “optimal” detection-vectors $|\Phi_{\text{GHZ}}\rangle$ and $|\Phi_{\text{W}}\rangle$ but also the two “optimal” measurement-settings Setting I. $(\sigma_x, \sigma_y, \sigma_z)$ and $(\sigma_z, -\sigma_y, \sigma_x)$, (by the transformation $\sigma_i \mapsto H\sigma_i H^\dagger$) which is equivalent to Setting II. (This equivalence holds only for *permutation-invariant* three-qubit states, when the three sets of observables $(X^{(1)}, Y^{(1)}, Z^{(1)})$ are the *same for each subsystem*. In this case one can check that the quantities

$\langle X_x^{(3)} \rangle^2 + \langle Y_x^{(3)} \rangle^2$ for $x = 0, 1, 2, 3$ are invariant under the transformation $(X^{(1)}, Y^{(1)}, Z^{(1)}) \leftrightarrow (Y^{(1)}, X^{(1)}, Z^{(1)})$ and $(X^{(1)}, Y^{(1)}, Z^{(1)}) \leftrightarrow (X^{(1)}, -Y^{(1)}, Z^{(1)})$. These can be seen by writing out the definitions given in Eqs. (25).)

We have tried some other randomly chosen detection-vectors which can be used to reduce the area where the criteria hold, and we get the same observations as at the end of the previous section: One can strenghtend the conditions only far from the $w = 0$ or $g = 0$ axes of the simplex, we have not found detection-vectors which give stronger condition for full-separability than the partial transposition criterion, and we have found settings by which the condition for biseparability can be strengthened, but these conditions are just a little bit stronger far from the axes than the ones in Section V D.

D. Criteria on matrix elements – a different approach

In Ref. [2] Gühne and Seevinck have given some further biseparability and full-separability criteria on the matrix elements:

$$\varrho \in \mathcal{D}_3^{2\text{-sep}} \implies |\varrho_{07}| \leq \sqrt{\varrho_{66}\varrho_{11}} + \sqrt{\varrho_{55}\varrho_{22}} + \sqrt{\varrho_{33}\varrho_{44}}, \quad (35a)$$

$$|\varrho_{12}| + |\varrho_{14}| + |\varrho_{24}| \leq \sqrt{\varrho_{00}\varrho_{33}} + \sqrt{\varrho_{00}\varrho_{55}} + \sqrt{\varrho_{00}\varrho_{66}} + (\varrho_{11} + \varrho_{22} + \varrho_{44})/2. \quad (35b)$$

The criterion (35a) is necessary and sufficient for GHZ-diagonal states and can also be obtained as a special case of the criteria of Section V B (Eq. (29)). However, this criterion—and the others in this Section—arises from a quite different approach as the one in (29), since these criteria have been derived from direct investigation of the matrix elements of pure separable states with the use of convexity argument. The criterion in (35b) is independent of the first one and it is quite strong in detection of W state mixed with white noise.

Of course these and the following inequalities can be written on local unitary transformed density matrices, and optimisation with regard to local unitaries might be necessary, but this can lead to very complicated expressions in the original matrix elements. An advantage of the method of the previous section is that it handles the matrix indices through the detection vector $|\Phi\rangle$.

The full-separability criteria of [2]:

$$\varrho \in \mathcal{D}_3^{3\text{-sep}} \implies |\varrho_{07}| \leq (\varrho_{11}\varrho_{22}\varrho_{33}\varrho_{44}\varrho_{55}\varrho_{66})^{1/6}, \quad (36a)$$

$$|\varrho_{12}| + |\varrho_{14}| + |\varrho_{24}| \leq \sqrt{\varrho_{00}\varrho_{33}} + \sqrt{\varrho_{00}\varrho_{55}} + \sqrt{\varrho_{00}\varrho_{66}}. \quad (36b)$$

(36a) is necessary and sufficient for GHZ state mixed with white noise, and (36b) is violated in the vicinity of

the W state. Moreover, one can obtain other conditions from (36a) by making substitutions as follows. Consider a fully separable pure state $\varrho^{3\text{-sep}} = |\Psi\rangle\langle\Psi|$, where $|\Psi\rangle = (a_0|0\rangle + a_1|1\rangle) \otimes (b_0|0\rangle + b_1|1\rangle) \otimes (c_0|0\rangle + c_1|1\rangle)$. Then the diagonal elements are $(\varrho^{3\text{-sep}})_{ijk,ijk} = |a_i|^2|b_j|^2|c_k|^2$, where we use the $ijk = 000, 001, \dots, 111$ binary indexing. Then

$$\begin{aligned} (\varrho^{3\text{-sep}})_{ijk,ijk} (\varrho^{3\text{-sep}})_{i'j'k',i'j'k'} &= (\varrho^{3\text{-sep}})_{ijk',ijk'} (\varrho^{3\text{-sep}})_{i'j'k,i'j'k} \\ &= (\varrho^{3\text{-sep}})_{ij'k,ij'k} (\varrho^{3\text{-sep}})_{i'jk',i'jk'} \\ &= (\varrho^{3\text{-sep}})_{i'jk,i'jk} (\varrho^{3\text{-sep}})_{ij'k',ij'k'}. \end{aligned}$$

Moreover, the rhs. of the inequality of (36a) can be written as $(\varrho_{11}^2 \varrho_{22}^2 \varrho_{33}^2 \varrho_{44}^2 \varrho_{55}^2 \varrho_{66}^2)^{1/12}$, and with these substitutions we can obtain a third power of four matrix elements under the 1/12th power. So we can get expressions of four matrix elements (e.g. $(\varrho_{11} \varrho_{22} \varrho_{44} \varrho_{77})^{1/4}$) on the rhs. of (36a). With the substitutions above one can get 28 different inequalities for (36a) with an expression of sixth order under the sixth root on the rhs. and 12 different ones with an expression of fourth order under the fourth root. For permutation-invariant states we have $\varrho_{11} = \varrho_{22} = \varrho_{44}$ and $\varrho_{66} = \varrho_{55} = \varrho_{33}$ and for our case $\varrho_{00} = \varrho_{77}$ as well, hence the number of different inequalities reduces to 8 and 5 respectively. The rhs's of inequality (36a) which are different for permutation-invariant matrices are as follows:

$$\begin{aligned} &(\varrho_{00}^3 \varrho_{77}^3)^{1/6}, (\varrho_{11} \varrho_{22} \varrho_{33} \varrho_{44} \varrho_{55} \varrho_{66})^{1/6}, \\ &(\varrho_{00} \varrho_{11} \varrho_{22} \varrho_{44} \varrho_{77}^2)^{1/6}, (\varrho_{00}^2 \varrho_{33} \varrho_{55} \varrho_{66} \varrho_{77})^{1/6}, \\ &(\varrho_{00} \varrho_{11}^2 \varrho_{66}^2 \varrho_{77})^{1/6}, (\varrho_{00}^2 \varrho_{11} \varrho_{66} \varrho_{77}^2)^{1/6}, \\ &(\varrho_{11}^2 \varrho_{22} \varrho_{44} \varrho_{66} \varrho_{77})^{1/6}, (\varrho_{00} \varrho_{11} \varrho_{33} \varrho_{55} \varrho_{66}^2)^{1/6} \end{aligned}$$

and

$$\begin{aligned} &(\varrho_{00}^2 \varrho_{77}^2)^{1/4}, \\ &(\varrho_{22} \varrho_{33} \varrho_{44} \varrho_{55})^{1/4}, (\varrho_{00} \varrho_{11} \varrho_{66} \varrho_{77})^{1/4}, \\ &(\varrho_{11} \varrho_{22} \varrho_{44} \varrho_{77})^{1/4}, (\varrho_{00} \varrho_{33} \varrho_{55} \varrho_{66})^{1/4}. \end{aligned}$$

It turns out that the strongest conditions can be given with the last of these and with the original one in (36a). (We could also make some substitutions in the rhs. of (36b) but these would not give stronger conditions than the original one.)

Writing out the criteria of biseparability and full separability we get:

$$\begin{aligned} \varrho \in \mathcal{D}_3^{2\text{-sep}} &\implies \tilde{g} \leq 3\sqrt{\tilde{d}(\tilde{d} + \tilde{w})}, \quad (37a) \\ &\tilde{w} \leq \sqrt{(\tilde{d} + \tilde{g})\tilde{d}} + (\tilde{d} + \tilde{w})/2 \quad (37b) \end{aligned}$$

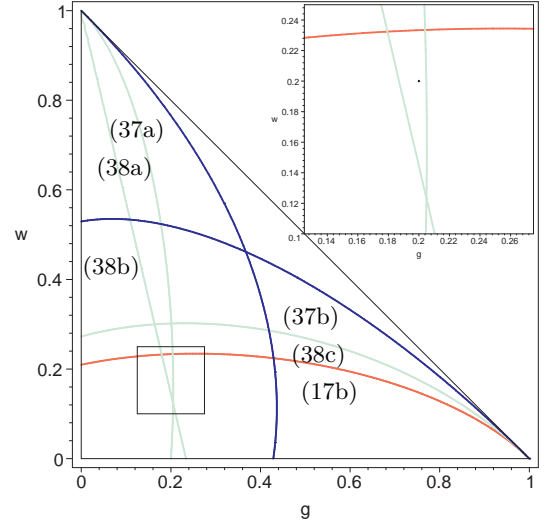


FIG. 8. (Color online) Criteria on matrix elements for state (4) on the g - w -plane. (Green/light grey curves: the borders of domains inside Eqs. (38a)-(38c) hold, blue/black curves: the borders of domains inside Eqs. (37a)-(37b) hold. Red/grey curve: the border of domain inside Eq. (17b) of partial transposition criterion hold, copied from Fig. 5. The point $g = 1/5$, $w = 1/5$ is also shown. The inequalities hold on the side of the curves containing the origin.)

and

$$\begin{aligned} \varrho \in \mathcal{D}_3^{3\text{-sep}} &\implies \tilde{g} \leq ((\tilde{d} + \tilde{w})\tilde{d})^{1/2}, \quad (38a) \\ &\tilde{g} \leq ((\tilde{d} + \tilde{g})\tilde{d}^3)^{1/4}, \quad (38b) \\ &\tilde{w} \leq \sqrt{(\tilde{d} + \tilde{g})\tilde{d}}. \quad (38c) \end{aligned}$$

(See in Eq. (A1) of the Appendix.) Clearly, the biseparability condition of (37a) is the same as the one of (32a) of the criterion on spin-observables but condition of (37b) is strictly stronger than the one of (32b). (On the $g = 0$ noisy W state it gives bound 9/17.) The full-separability condition of (38a) is the same as the one of (33a) of the criterion on spin-observables (and the one of (17a) of partial transposition criterion as well) but the condition of (38c) is weaker than the one of (33b) of the criterion on spin-observables. Hence at this point these criteria are stronger for biseparability but weaker for full separability than the criteria on spin-observables for our state. But we have another full-separability condition: (38b) can be stronger in a region than the ones based on the partial transposition criterion. The states of parameters in this region are entangled ones of positive partial transpose, no pure state entanglement can be distilled from them. The borders of the domains in which these conditions hold and the region of PPTESs can be seen in Fig. 8.

Now we show a representing matrix of the region of PPTESs determined by (17a), (17b) and (38b). One can check that the state of parameters $g = 1/5$, $w = 1/5$ is contained by this set and the explicit form of (4) for this

point is

$$\varrho^{\text{PPTES}} = \frac{1}{120} \begin{bmatrix} 21 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 12 \\ \cdot & 17 & 8 & \cdot & 8 & \cdot & \cdot & \cdot \\ \cdot & 8 & 17 & \cdot & 8 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 9 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 8 & 8 & \cdot & 17 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 9 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 9 & \cdot \\ 12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 21 \end{bmatrix}. \quad (39)$$

VI. SLOCC CLASSES OF GENUINE THREE-QUBIT ENTANGLEMENT

A fully entangled three-qubit *pure* state can be either of GHZ-type or of W-type [14] in the sense of *Stochastic Local Operations and Classical Communications* (SLOCC, [13]): vectors of these two different types can not be transformed into each other by local invertible operations. These fully entangled vectors $|\psi\rangle$ can be classified by the so called *three-tangle* $\tau_{123}(\psi)$ [15] as $\tau_{123}(\psi) \neq 0$ exactly for the GHZ-type vectors.

In Ref. [12] Acín *et. al.* have investigated the classification of *mixed* three-qubit states in the sense of SLOCC and they have shown that Class 1 of fully entangled states can be divided into two subsets, namely the ones of GHZ and W-type entanglement, by the following definitions. A state is of *W-type* (\mathcal{D}_3^W) if it can be expressed as a mixture of projectors onto 2-separable and W-type vectors (therefore \mathcal{D}_3^W is also a convex set) and *GHZ-type* vector is required for a GHZ-type mixed state. Hence the following holds:

$$\mathcal{D}_3^{3\text{-sep}} \subset \mathcal{D}_3^{2\text{-sep}} \subset \mathcal{D}_3^W \subset \mathcal{D}_3^{\text{GHZ}} \equiv \mathcal{D}_3^{1\text{-sep}} \equiv \mathcal{D}_3. \quad (40)$$

Let *Class W* the set $\mathcal{D}_3^W \setminus \mathcal{D}_3^{2\text{-sep}}$ and *Class GHZ* the set $\mathcal{D}_3^{\text{GHZ}} \setminus \mathcal{D}_3^W$, so Class 1 = Class W \cup Class GHZ. The mixed state extension of τ_{123}

$$\tau_{123}(\varrho) = \min \left\{ \sum_i p_i \tau_{123}(\psi_i) \mid \sum_i p_i |\psi_i\rangle\langle\psi_i| = \varrho \right\} \quad (41)$$

is a good indicator for Class GHZ: $\tau_{123}(\varrho) \neq 0$ exactly for Class GHZ.

A method to determine to which class a given mixed state belongs is the use of witness operators. [8] A hermitian operator W is a *witness operator* for a convex compact set $\mathcal{C} \subset \mathcal{D}_N$ if $0 \leq \text{Tr } W\sigma$ for all $\sigma \in \mathcal{C}$ and there exists $\varrho \notin \mathcal{C}$ for which $\text{Tr } W\varrho < 0$. Hence the negativity of expectation value of the observable W bears witness that the state does not belong to \mathcal{C} . In Ref. [12] there have been given some witnesses for \mathcal{D}_3^W and $\mathcal{D}_3^{\text{GHZ}}$:

$$W_{\text{GHZ}} = \frac{3}{4}\mathbb{I} - |\text{GHZ}\rangle\langle\text{GHZ}| \quad (42)$$

can detect $\mathcal{D}_3^{\text{GHZ}}$ and

$$W_{W_1} = \frac{2}{3}\mathbb{I} - |W\rangle\langle W|, \quad (43a)$$

$$W_{W_2} = \frac{1}{2}\mathbb{I} - |\text{GHZ}\rangle\langle\text{GHZ}| \quad (43b)$$

can detect \mathcal{D}_3^W . With these we have

$$\begin{aligned} \varrho \in \mathcal{D}_3^W &\implies \\ 0 \leq \text{Tr } W_{\text{GHZ}}\varrho &= (20\tilde{d} - 2\tilde{g} + 9\tilde{w})/4 \\ &= (5 - 7g + w)/8 \end{aligned} \quad (44)$$

and if the inequality is violated then $\varrho \in \text{Class GHZ}$, as well as

$$\begin{aligned} \varrho \in \mathcal{D}_3^{2\text{-sep}} &\implies \\ 0 \leq \text{Tr } W_{W_1}\varrho &= (13\tilde{d} + 4\tilde{g} - 3\tilde{w})/3 \\ &= (13 + 3g - 21w)/24, \end{aligned} \quad (45a)$$

$$\begin{aligned} 0 \leq \text{Tr } W_{W_2}\varrho &= (6\tilde{d} - 2\tilde{g} + 3\tilde{w})/2 \\ &= (3 - 7g + w)/8 \end{aligned} \quad (45b)$$

and if either or both of the inequalities is violated then $\varrho \in \text{Class 1}$. In Fig. 9 we plot the lines on which these inequalities are saturated. It can be checked that (45a) and (45b) gives weaker condition for biseparability than (37a) and (37b) of the previous Section. We can conclude that all the states in the blue/grey domain belong to Class GHZ, and the biseparable states are enclosed by the blue/black curves, however, both type of fully entangled states can be here too.

The equality in (44) gives an “upper bound” for the border of Class GHZ. (See blue/grey domain in Fig. 9.) Fortunately, we have a possibility to give also a “lower bound” for that, thanks to the results of Lohmayer *et. al.* [19]. They have studied the GHZ-W mixture ($d = 0$) and they have found that there exists a decomposition of projectors onto vectors of vanishing three-tangle if and only if $0 \leq g \leq g_0 = 4 \cdot 2^{1/3}/(3 + 4 \cdot 2^{1/3}) = 0.626851\dots$, hence for these parameters the mixed state extension (41) of the three-tangle is zero. If we mix the states of this interval with white noise then the three-tangle remains zero and neither of these states can belong to Class GHZ. So we can state that

$$\varrho \in \mathcal{D}_3^{\text{GHZ}} \implies w < \frac{3}{4 \cdot 2^{1/3}}g. \quad (46)$$

which holds under the green/light grey line of Fig. 9. This condition is quite weak, but we can make it stronger. Recall that on the $w = 0$ line (noisy GHZ state) $\varrho \in \text{Class 1}$ if and only if $3/7 < g \leq 1$. (See Section III.) So the convexity of \mathcal{D}_3^W restricts Class GHZ to be inside the triangle defined by the vertices ($g = 3/7, w = 0$), ($g = 1, w = 0$) and ($g = g_0, w = 1 - g_0$). (Union of tinted domains in Fig. 9.) So we can conclude that all the states in the blue/grey domain belong to Class GHZ, and the border of Class GHZ is in the red/light grey domain of Fig. 9.

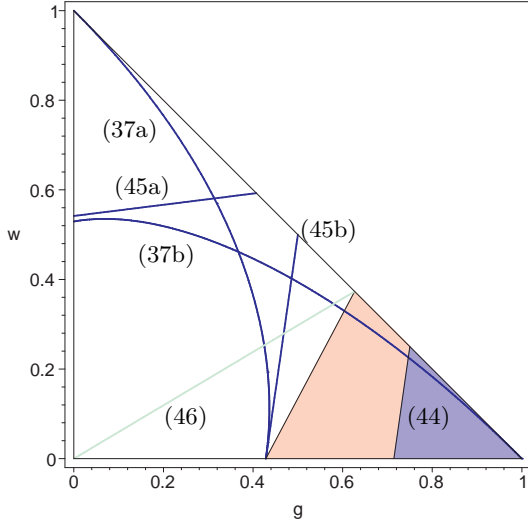


FIG. 9. (Color online) Criteria on three-partite entanglement classes for state (4) on the g - w -plane. (Blue/black straight lines: the borders of domains inside Eqs. (45a)-(45b) hold, blue/black curves of second order: the borders of domains inside Eqs. (37a)-(37b) hold, copied from Fig. 8. Eq. (44) holds inside the blue/grey domain, and the border of Class GHZ is inside the red/light grey domain. The inequalities hold on the side of the curves containing the origin. Eq. (46) holds under the green/light grey line.)

VII. CONCLUSIONS

In this paper we have investigated the noisy GHZ-W mixture and demonstrated some necessary but not sufficient criteria for different classes of separability. With these criteria we can restrict these classes into some domains of the 2-dimension simplex. It has turned out that the strongest conditions was (17a), (38b) and (17b) for full separability, (17a) and (17b) for the union of Classes 2.8 and 3 and (37a) and (37b) for biseparability. These have been obtained from the partial transposition criterion of Peres [1] and the criteria of Gühne and Seevinck [2] dealing with matrix elements. Only these latter criteria have turned out to be strong enough to reveal a set of entangled states of positive partial transpose. (The set of these states can be given by the conditions of (17a), (17b) and (38b). An example is given in Eq. (39).)

Besides this, some remarkable coincidences have also appeared: some parts of some bipartite separability criteria have proved to be necessary and sufficient for separability classes of the GHZ-white noise mixture. (e.g. the majorisation criterion and the entropy criterion in the $\alpha \rightarrow \infty$ limit as well, and reduction criterion.) This is interesting because e.g. the majorisation criterion—as our knowledge—does not state anything about a density matrix which is majorized by only either of its subsystems. We do not think that this would be more than a coincidence, however, the GHZ state is very special so it is an interesting question whether this can be generalized to the n -qubit noisy GHZ state.

Another interesting observation was that the two settings/measurement vectors strong in detection of GHZ and W state are related by local unitary Hadamard transformation in the criteria on spin-observables of Seevinck and Uffink (Section VB) and also in the criteria of Gabriel *et. al.* (Section VC). The transformation on settings/measurement vectors can also be written on the state $\varrho \mapsto (H^{\otimes 3})^\dagger \varrho H^{\otimes 3}$, which means that we can use the same measurements on the transformed state for the detection of W state as for the detection of GHZ state.

We have also investigated the SLOCC classes of fully entangled states and we have given restrictions for Class GHZ.

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Appendix A: Matrices

In this Appendix we show the density matrix ϱ given in Eq. (4), its partial transpose, its reshufflings, and its marginals explicitly. Note that we use the renormalized parameters $\tilde{d} = d/8$, $\tilde{g} = g/2$, $\tilde{w} = w/3$ and the constraint $d + g + w = 1$ holds.

$$\varrho = \begin{bmatrix} \tilde{d} + \tilde{g} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{g} \\ \cdot & \tilde{d} + \tilde{w} & \tilde{w} & \cdot & \tilde{w} & \cdot & \cdot \\ \cdot & \tilde{w} & \tilde{d} + \tilde{w} & \cdot & \tilde{w} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \tilde{d} & \cdot & \cdot & \cdot \\ \cdot & \tilde{w} & \tilde{w} & \cdot & \tilde{d} + \tilde{w} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{d} & \cdot \\ \tilde{g} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{d} + \tilde{g} \end{bmatrix}, \quad (\text{A1})$$

$$\varrho^{T_1} = \begin{bmatrix} \tilde{d} + \tilde{g} & \cdot & \cdot & \cdot & \cdot & \tilde{w} & \tilde{w} & \cdot \\ \cdot & \tilde{d} + \tilde{w} & \tilde{w} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \tilde{w} & \tilde{d} + \tilde{w} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \tilde{d} & \tilde{g} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \tilde{g} & \tilde{d} + \tilde{w} & \cdot & \cdot & \cdot \\ \tilde{w} & \cdot & \cdot & \cdot & \cdot & \tilde{d} & \cdot & \cdot \\ \tilde{w} & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{d} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \tilde{d} + \tilde{g} \end{bmatrix}, \quad (\text{A2})$$

$$\varrho^{23} = \begin{bmatrix} 2\tilde{d} + \tilde{g} + \tilde{w} & \cdot & \cdot & \cdot \\ \cdot & 2\tilde{d} + \tilde{w} & \tilde{w} & \cdot \\ \cdot & \tilde{w} & 2\tilde{d} + \tilde{w} & \cdot \\ \cdot & \cdot & \cdot & 2\tilde{d} + \tilde{g} \end{bmatrix}, \quad (\text{A3})$$

It can be easily checked that its partial transpose is not positive, so it is not α_2 -separable, (the partial transposition criterion see in Section IV C) hence it is in Class 2.1.

An example for a permutation-invariant state in Class 2.8 is given in Eq. (14) of [12] with $a = b = \frac{1}{c}$:

$$\varrho^{\text{Class 2.8}} = \frac{1}{2 + 3\left(a + \frac{1}{a}\right)} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{a} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{a} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{a} & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}, \quad (\text{B2})$$

where $0 < a$. This state is entangled if and only if $a \neq 1$, and $\varrho^{\text{Class 2.8}} \in \mathcal{D}_3^{\alpha_2}$ for all α_2 . [12]

Appendix C: Wootters concurrence

For mixed states the only measure of entanglement known explicitly is the *Wootters-concurrence* of two qubits: [17, 18]

$$\mathcal{C}(\omega) = \max\{0, \lambda_1^\downarrow - \lambda_2^\downarrow - \lambda_3^\downarrow - \lambda_4^\downarrow\}, \quad (\text{C1})$$

where ω is a two-qubit density matrix and λ_i^\downarrow s are the eigenvalues of the positive matrix $\sqrt{\sqrt{\omega}\tilde{\omega}\sqrt{\omega}}$ in non-increasing order. Here $\tilde{\omega}$ denotes the *spin-flipped* density matrix: $\tilde{\omega} = (\sigma_2 \otimes \sigma_2)\omega^*(\sigma_2 \otimes \sigma_2)$ where $\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and ω^* denotes the complex conjugation. Equivalently λ_i s can be calculated as the square roots of the positive eigenvalues of the matrix $\tilde{\omega}\omega$.

Let us calculate now the concurrence given in Eq. (C1) of the matrix ϱ^{23} . (See in Eq. (A3) of the Appendix.) Since the spin-flip in the two-qubit case means transpose with respect to the antidiagonal then multiplication of neither diagonal nor antidiagonal entries by -1 , one can easily get:

$$\begin{aligned} \text{Spect}(\tilde{\varrho}^{23}\varrho^{23}) = \\ \{ & 4(\tilde{d} + \tilde{w})^2 = 4(3 - 3g + 5w)^2/24^2, \\ & (2\tilde{d} + \tilde{g})(2\tilde{d} + \tilde{g} + \tilde{w}) \\ & = 12(1 + g - w)(3 + 3g + w)/24^2, \\ & (2\tilde{d} + \tilde{g})(2\tilde{d} + \tilde{g} + \tilde{w}) \\ & = 12(1 + g - w)(3 + 3g + w)/24^2, \\ & 4\tilde{d}^2 = 36(1 - g - w)^2/24^2 \} \end{aligned} \quad (\text{C2})$$

Clearly, the last eigenvalue is the smallest one. If $4(\tilde{d} + \tilde{w})^2 \leq (2\tilde{d} + \tilde{g})(2\tilde{d} + \tilde{g} + \tilde{w})$ then $\lambda_1^\downarrow = \lambda_2^\downarrow$ hence $\mathcal{C}(\varrho^{23}) = 0$. If $4(\tilde{d} + \tilde{w})^2 \geq (2\tilde{d} + \tilde{g})(2\tilde{d} + \tilde{g} + \tilde{w})$, then $\lambda_2^\downarrow = \lambda_3^\downarrow$ and $\mathcal{C}(\varrho^{23})$ can be nonzero. It turns out that

$$\begin{aligned} \mathcal{C}(\varrho^{23}) &= 2\tilde{w} - 2\sqrt{(2\tilde{d} + \tilde{g})(2\tilde{d} + \tilde{g} + \tilde{w})} \\ &= \frac{2}{3}w - \frac{1}{2\sqrt{3}}\sqrt{(1 + g - w)(3 + 3g + w)} \end{aligned} \quad (\text{C3})$$

if $0 \leq \tilde{w}^2 - (2\tilde{d} + \tilde{g})(2\tilde{d} + \tilde{g} + \tilde{w}) = (-9g^2 + 19w^2 + 6gw - 18g + 6w - 9)/12^2$, otherwise $\mathcal{C}(\varrho^{23}) = 0$. (Fig. 10) It takes its maximum $2/3$ in $g = 0$, $w = 1$, i.e. in pure W-state. For the GHZ-W mixture ($d = 0$) we get back the result of Ref. [19].

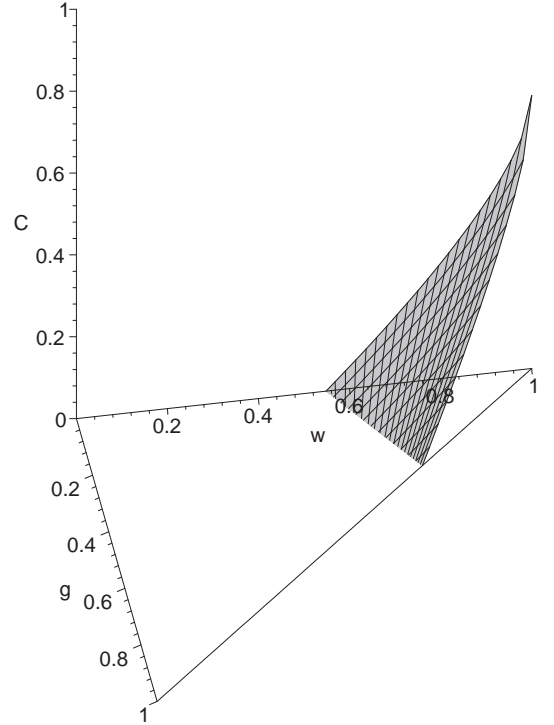


FIG. 10. Wootters-concurrence of ϱ^{23} on the g - w -plane. (Eq. (C3))

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